Laplace Transform Time Response Utility

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Asian IBIS Summit
Shanghai, China
November 9, 2015

(Presented by Anders Ekholm, Ericsson)
Originally presented May 13, 2015

Some Applications

- **Show step and impulse response for network analysis**
- **Show step and impulse response for lower-order, reduced order (or pole-zero) Touchstone formulations in IBIS-AMI analysis**
- **Embed for time-response displays in analysis applications by inserting calculations at top**

Notation and Introduction

Laplace Transform	$X(s) = \dfrac{a_{n-1}s^{n-1} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_0},$
Differential Equation	$x^n(t) + b_{n-1}x^{n-1}(t) + \cdots + b_0 x(t) = 0$ initial conditions, $x(0), \cdots, x^{n-1}(0),$

Utility calculates and displays immediately 101 points for $x^i(t)$, i=0 to i=26 for the time response and all of its derivatives

Extended for more time points by copying and pasting last row

Can be used as an embedded utility involving other Laplace Transform calculations

Enter Laplace Transform Num. and Den. Coefficients and Time-Step

Laplace Transform Numerator and Denominator Coefficients

a7	a6	a5	a4	a3	a2	a1	a0
0	1	-21	210	-1260	4725	-10395	10395
b7	**b6**	**b5**	**b4**	**b3**	**b2**	**b1**	**b0**
1	21	210	1260	4725	10395	10395	0
	T-Step s						
Select	0.08						

Step Response of 6th order Bessel (maximally flat envelope delay, MFED) all-pass function

Change time-step to zoom-in or zoom-out and to change resolution

The graph auto-scales over 101 points

6^{th} Order MFED All-Pass Step Response

Laplace Transform Numerator and Denominator Coefficients

a7	a6	a5	a4	a3	a2	a1	a0
0	1	-21	210	-1260	4725	-10395	10395
b7	**b6**	**b5**	**b4**	**b3**	**b2**	**b1**	**b0**
1	21	210	1260	4725	10395	10395	0

	T-Step s
Select	0.08

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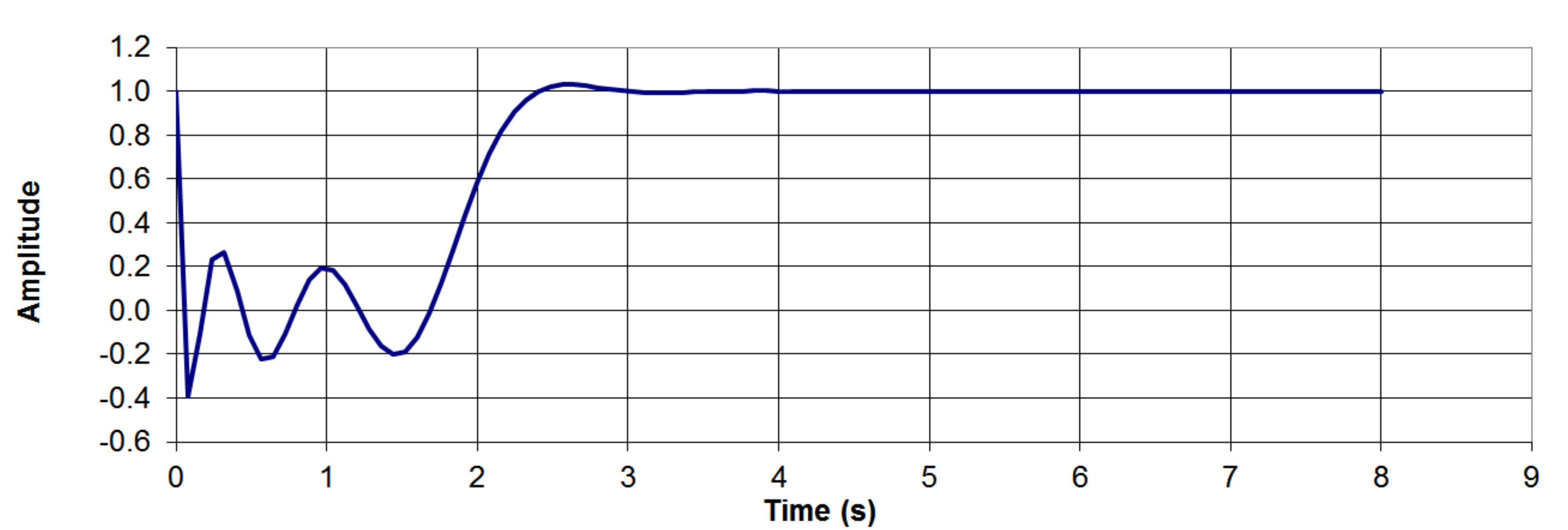
a6	a5	a4	a3	a2	a1	a0
1	-21	210	-1260	4725	-10395	10395
b6	**b5**	**b4**	**b3**	**b2**	**b1**	**b0**
21	210	1260	4725	10395	10395	0



6^{th} Order MFED Low-Pass Step Response Input

Laplace Transform Numerator and Denominator Coefficients							
a7	a6	a5	a4	a3	a2	a1	a0
0							10395
b7	b6	b5	b4	b3	b2	b1	b0
1	21	210	1260	4725	10395	10395	0
	T-Step s						
Select	0.08						

Convert all-pass to low-pass filter by zeroing out numerator coefficients (click/back-space or enter 0) for real-time modification

6^{th} Order MFED Low-Pass Step Response

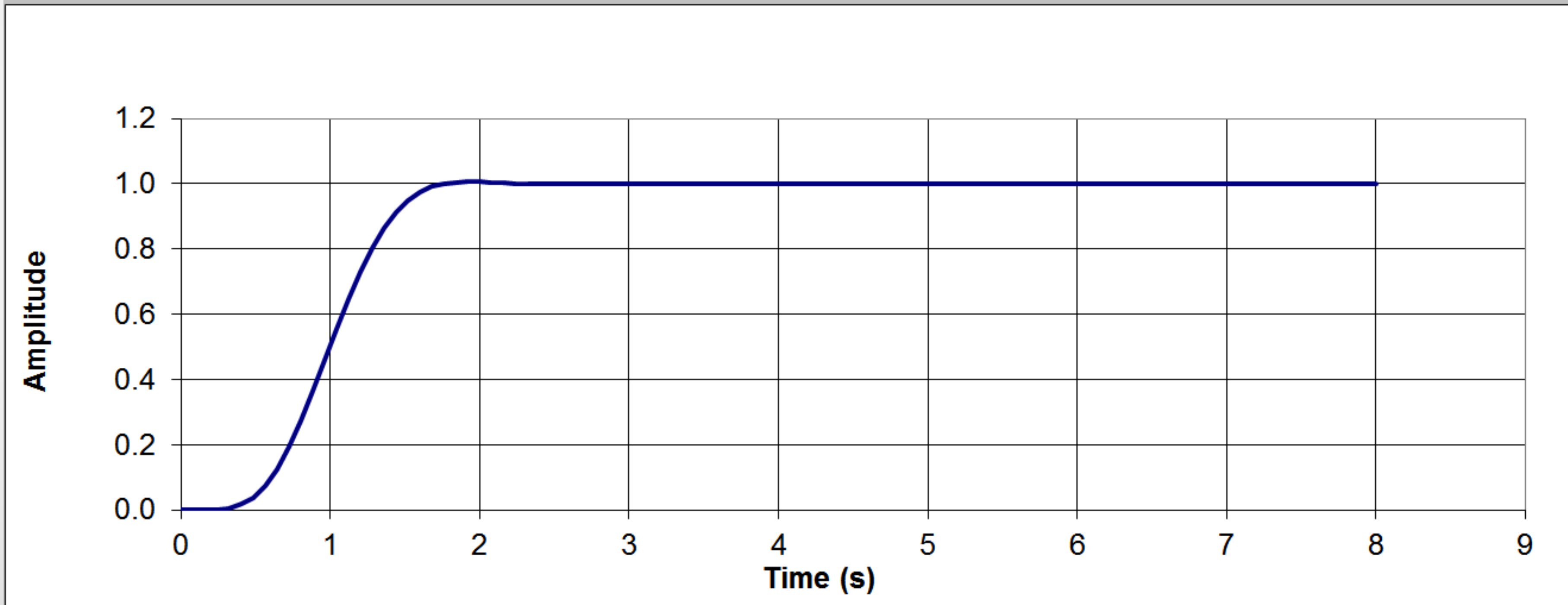
Laplace Transform Numerator and Denominator Coefficients

a7	a6	a5	a4	a3	a2	a1	a0
0							10395
b7	**b6**	**b5**	**b4**	**b3**	**b2**	**b1**	**b0**
1	21	210	1260	4725	10395	10395	0

	T-Step s
Select	0.08

Copy for Pasting

a6	a5	a4	a3	a2	a1	a0
1	-21	210	-1260	4725	-10395	10395
b6	**b5**	**b4**	**b3**	**b2**	**b1**	**b0**
21	210	1260	4725	10395	10395	0



7^{th} Order MFED Low-Pass Impulse Response Input

Laplace Transform Numerator and Denominator Coefficients

a7	a6	a5	a4	a3	a2	a1	a0
0	0	0	0	0	0	0	135135
b7	b6	b5	b4	b3	b2	b1	b0
1	28	378	3150	17325	62370	135135	135135

	T-Step s
Select	0.025

7^{th} Order MFED Low-Pass Impulse Response

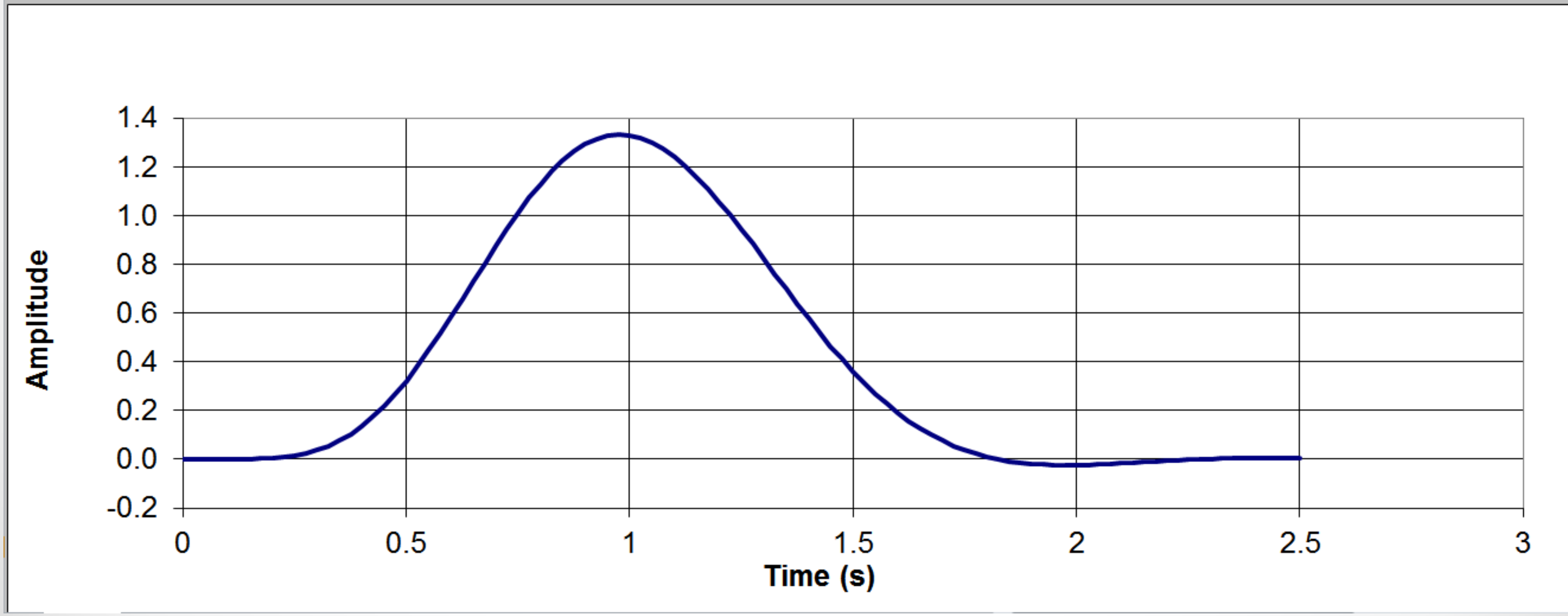
Laplace Transform Numerator and Denominator Coefficients

a7	a6	a5	a4	a3	a2	a1	a0
0	0	0	0	0	0	0	135135
b7	**b6**	**b5**	**b4**	**b3**	**b2**	**b1**	**b0**
1	28	378	3150	17325	62370	135135	135135

	T-Step s
Select	0.025

Copy for Pasting

a6	a5	a4	a3	a2	a1	a0
1	-21	210	-1260	4725	-10395	10395
b6	**b5**	**b4**	**b3**	**b2**	**b1**	**b0**
21	210	1260	4725	10395	10395	0



$X(s) = 1/(s^2 + 1)$ Sine Wave with Left-Shifted Coefficients

Laplace Transform Numerator and Denominator Coefficients

a7	a6	a5	a4	a3	a2	a1	a0
0	0	1					
b7	b6	b5	b4	b3	b2	b1	b0
1	0	1					

	T-Step s
Select	0.2617994

Set Time-Step =PI()/12 for exact $\pi/12$ (15 degree) steps

Can compare response with exact solution: $x(t) = \sin(\pi t/12)$

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Sine Wave Response

Laplace Transform Numerator and Denominator Coefficients

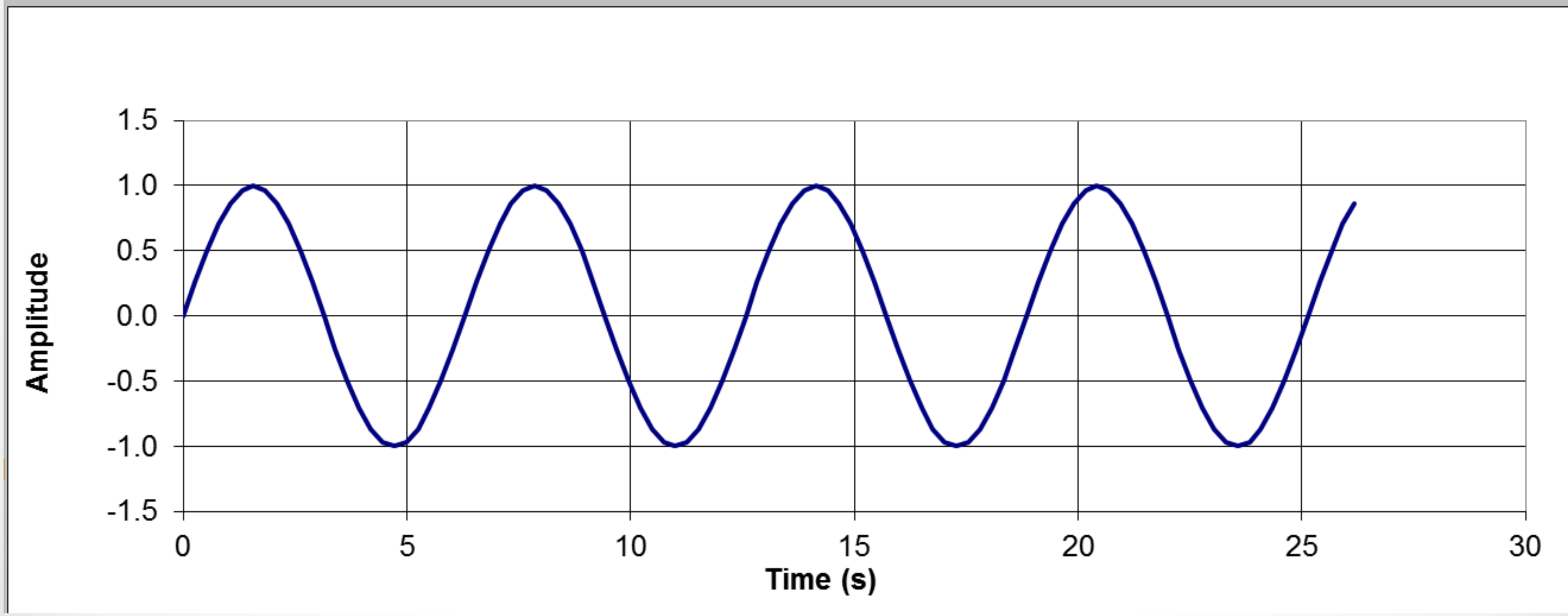
a7	a6	a5	a4	a3	a2	a1	a0
0	0	1					
b7	b6	b5	b4	b3	b2	b1	b0
1	0	1					

T-Step s

Select 0.2617994

Copy for Pasting

a6	a5	a4	a3	a2	a1	a0
1	-21	210	-1260	4725	-10395	10395
b6	b5	b4	b3	b2	b1	b0
21	210	1260	4725	10395	10395	0



Recursive Taylor Series Method (Repeat b and c)

a) Initialize: *i* = 1, … , *n*-1 (n = 7)

$$x(0) = a_{n-1} \qquad x^{i}(0) = a_{n-1-i} - \sum_{j=0}^{i-1} b_{n-i-j}\, x^{j}(0)$$

b) Extend: *i* = *n*, … , *p* (p = 26)

$$x^{i}(t) = -\sum_{j=0}^{n-1} b_{j}\, x^{i-n-j}(t)$$

c) Next time step: *i* = 0, … , *n*-1 (Taylor Series)

$$x^{i}(t+T) = \sum_{j=i}^{p} x^{j}(t) \frac{T^{j-i}}{(j-i)!}$$

R. I. Ross, "Evaluating the Transient Response of a Network Function," *Proc. IEEE*, vol.55, pp. 615-616, May 1967

Spread Sheet Details

Laplace Transform Numerator and Denominator Coefficients								Copy for Pastin			
a7	a6	a5	a4	a3	a2	a1	a0	a6	a5	a4	a3
0	0	1						1	-21	210	-1260
b7	b6	b5	b4	b3	b2	b1	b0	b6	b5	b4	b3
1	0	1						21	210	1260	4725

	T-Step s
Select	0.2617994

(a) Initialize

Copy to Reset Sheet

N	0	1	2	3	4	5	6	7	8	9	10
T^n/n!	1	0.2617994	0.0342695	0.002990575	0.000195733	1.02485E-05	4.47177E-07	1.67244E-08	5.47304E-10	1.59204E-11	4.16796E-13

$V^{(N)}(t)$ / t	0	1	2	3	4	5	6	7	8	9	10
0	0.0000	1.000E+00	0.000E+00	-1.000E+00	0.000E+00	1.000E+00	0.000E+00	-1.000E+00	0.000E+00	1.000E+00	0.000E+00
0.2617994	0.2588	9.659E-01	-2.588E-01	-9.659E-01	2.588E-01	9.659E-01	-2.588E-01	-9.659E-01	2.588E-01	9.659E-01	-2.588E-01
0.5235988	0.5000	8.660E-01	-5.000E-01	-8.660E-01	5.000E-01	8.660E-01	-5.000E-01	-8.660E-01	5.000E-01	8.660E-01	-5.000E-01
0.7853982	0.7071	7.071E-01	-7.071E-01	-7.071E-01	7.071E-01	7.071E-01	-7.071E-01	-7.071E-01	7.071E-01	7.071E-01	-7.071E-01
1.0471976	0.8660	5.000E-01	-8.660E-01	-5.000E-01	8.660E-01	5.000E-01	-8.660E-01	-5.000E-01	8.660E-01	5.000E-01	-8.660E-01
1.3089969	0.9659	2.588E-01	-9.659E-01	-2.588E-01	9.659E-01	2.588E-01	-9.659E-01	-2.588E-01	9.659E-01	2.588E-01	-9.659E-01
1.5707963	1.0000	7.342E-16	-1.000E+00	-7.342E-16	1.000E+00	7.342E-16	-1.000E+00	-7.342E-16	1.000E+00	7.342E-16	-1.000E+00
1.8325957	0.9659	-2.588E-01	-9.659E-01	2.588E-01	9.659E-01	-2.588E-01	-9.659E-01	2.588E-01	9.659E-01	-2.588E-01	-9.659E-01
2.0943951	0.8660	-5.000E-01	-8.660E-01	5.000E-01	8.660E-01	-5.000E-01	-8.660E-01	5.000E-01	8.660E-01	-5.000E-01	-8.660E-01
2.3561945	0.7071	-7.071E-01	-7.071E-01	7.071E-01	7.071E-01	-7.071E-01	-7.071E-01	7.071E-01	7.071E-01	-7.071E-01	-7.071E-01
2.6179939	0.5000	-8.660E-01	-5.000E-01	8.660E-01	5.000E-01	-8.660E-01	-5.000E-01	8.660E-01	5.000E-01	-8.660E-01	-5.000E-01

(c) x(t+T) Taylor Series for new row

(b) Extend thru N=26

Accurate Time Response for $X(s) = 1/(s^2 + 1); x(t) = \sin(\pi t/12)$

	X	Y	Z	AA	AB	AC
103	-1.05412673E-14	1.00000000E+00	1.05412673E-14	-1.00000000E+00	-1.05412673E-14	1.00000000E+00
104	2.58819045E-01	9.65925826E-01	-2.58819045E-01	-9.65925826E-01	2.58819045E-01	9.65925826E-01
105	5.00000000E-01	8.66025404E-01	-5.00000000E-01	-8.66025404E-01	5.00000000E-01	8.66025404E-01
106	7.07106781E-01	7.07106781E-01	-7.07106781E-01	-7.07106781E-01	7.07106781E-01	7.07106781E-01
107	8.66025404E-01	5.00000000E-01	-8.66025404E-01	-5.00000000E-01	8.66025404E-01	5.00000000E-01
108	9.65925826E-01	2.58819045E-01	-9.65925826E-01	-2.58819045E-01	9.65925826E-01	2.58819045E-01
109	1.00000000E+00	1.10769747E-14	-1.00000000E+00	-1.10769747E-14	1.00000000E+00	1.10769747E-14
110	9.65925826E-01	-2.58819045E-01	-9.65925826E-01	2.58819045E-01	9.65925826E-01	-2.58819045E-01
111	8.66025404E-01	-5.00000000E-01	-8.66025404E-01	5.00000000E-01	8.66025404E-01	-5.00000000E-01
112	7.07106781E-01	-7.07106781E-01	-7.07106781E-01	7.07106781E-01	7.07106781E-01	-7.07106781E-01
113	5.00000000E-01	-8.66025404E-01	-5.00000000E-01	8.66025404E-01	5.00000000E-01	-8.66025404E-01

Iterative calculation = exact response to 9 digits

- **up to 101 data points**
- **up to the 26th derivative**

(Table resolution increased to 9 digits to show accuracy)

5^{th}-order Step Response

Laplace Transform Numerator and Denominator Coefficients							
a7	a6	a5	a4	a3	a2	a1	a0
0				-4	4		
b7	b6	b5	b4	b3	b2	b1	b0
1	6	13	12	4	0		
	T-Step s						
Select	0.25						

$X(s) = 4(-s + 1)/[(s + 1)^2 (s + 2)^2 s] = (-4s + 4)/(s^5 + 6s^4 + 13s^3 + 12s^2 + 4s)$

Laplace transform is normalized ($b_7 = 1$)
Left-shift the numerator and denominator coefficients
Step response means b_2 is 0
Right-hand plane zero creates pre-shoot

5th-Order Step Response

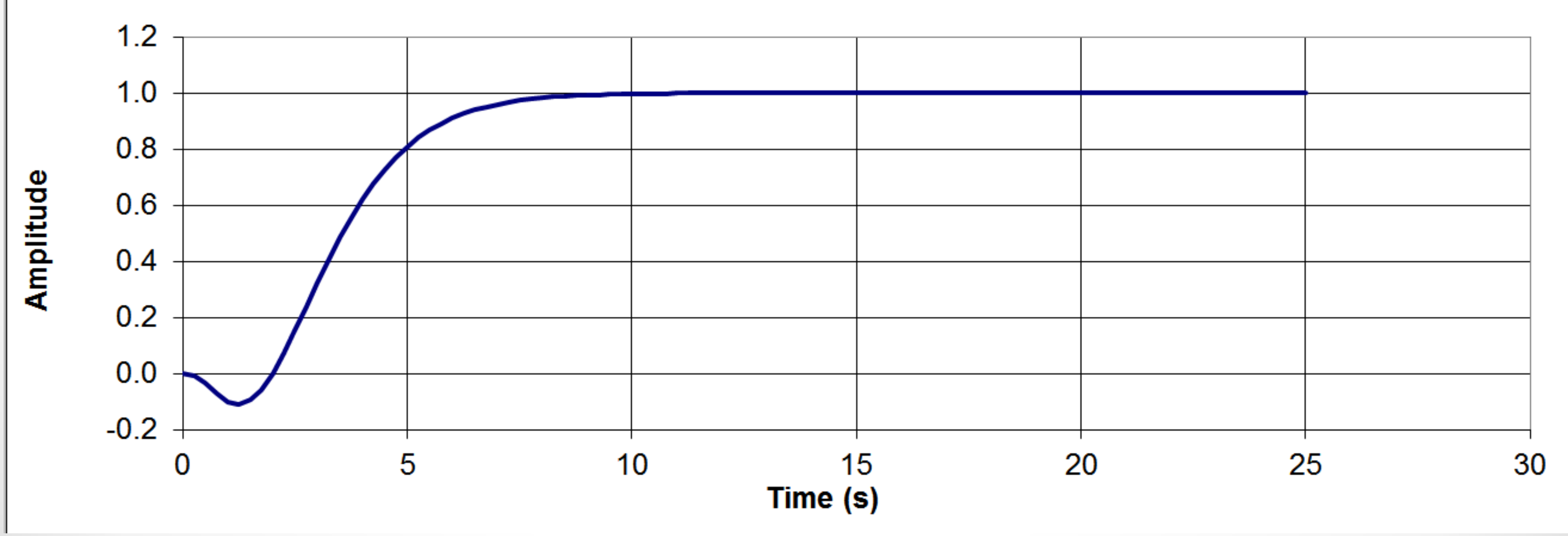
Laplace Transform Numerator and Denominator Coefficients

a7	a6	a5	a4	a3	a2	a1	a0
0				-4	4		
b7	**b6**	**b5**	**b4**	**b3**	**b2**	**b1**	**b0**
1	6	13	12	4	0		

	T-Step s
Select	0.25

Copy for Pasting

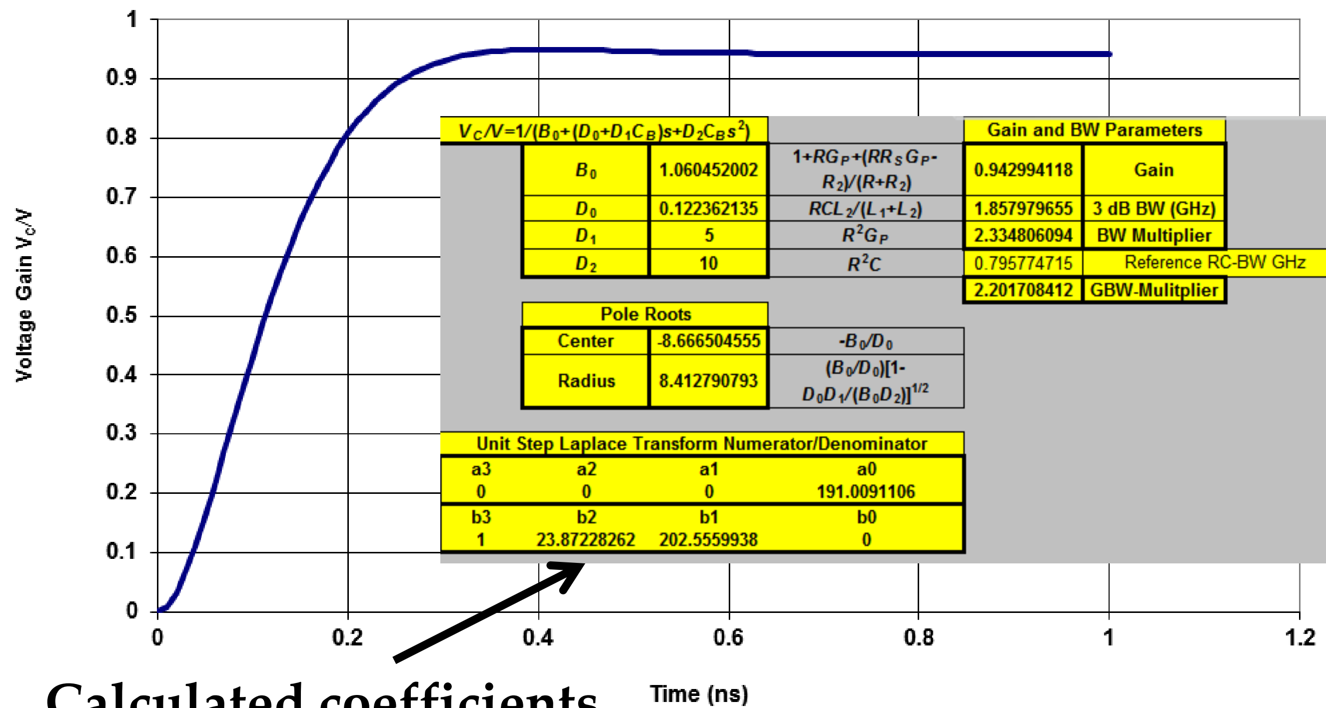
a6	a5	a4	a3	a2	a1	a0
1	-21	210	-1260	4725	-10395	10395
b6	**b5**	**b4**	**b3**	**b2**	**b1**	**b0**
21	210	1260	4725	10395	10395	0



Embedded with Constant-R Bridged T-Coil Calculations

Bridged T-Coil Input		
R	50	
C (pF)	4	C (nF) 0.004
R_S	10	
G_P $(1/R_P)$	0.002	R_p 500
R_1	1.9073	
T-step (ns)	0.01	
Theta (deg)	33	

Calculated Values		Value, Formulas
C_B (pF)	0.523535237	C_B (nF) 0.000523535
R_1	1.9073	Choose R_1 above so that R_2 >= 0
R_2 (Check R_2>=0)	3.099998636	$R_2=[R^2G_P-R_1(1+G_P(R_S+R))]/[1+G_P(R_1+R_S-R)]$
L_1+L_2 (nH)	10.01459727	$\{(R-R_1)^2/[1+G_P(R_1+R_S-R)]\}C$
L_1 (nH)	3.887559745	$(R_1-R)(R_1+R_S-R)/\{1+[(1+G_P(R_1+R_S-R)]+[1+G_P(R_1+R_S-R)]^{1/2}\}*C$
L_2 (nH)	6.127037527	
L_3 (nH)	-1.069612463	$R^2C_B-L_1L_2/(L_1+L_2)$



$V_C/V=1/(B_0+(D_0+D_1C_B)s+D_2C_Bs^2)$		
B_0	1.060452002	$1+RG_P+(RR_SG_P-R_2)/(R+R_2)$
D_0	0.122362135	$RCL_2/(L_1+L_2)$
D_1	5	R^2G_P
D_2	10	R^2C

Gain and BW Parameters	
0.942994118	Gain
1.857979655	3 dB BW (GHz)
2.334806094	BW Multiplier
0.795774715	Reference RC-BW GHz
2.201708412	GBW-Mulitplier

Pole Roots		
Center	-8.666504555	$-B_0/D_0$
Radius	8.412790793	$(B_0/D_0)[1-D_0D_1/(B_0D_2)]^{1/2}$

Unit Step Laplace Transform Numerator/Denominator			
a3	a2	a1	a0
0	0	0	191.0091106
b3	b2	b1	b0
1	23.87228262	202.5559938	0

Calculated coefficients

Scaled Time (ns), L (nH), C (nF) with 3rd order Laplace Transform sheet

Closed-form equations inserted above Time Response

Guidance

- Normalize coefficients (highest order denominator coefficient set to 1)
- Scale coefficients so that values are meaningful for time-steps between 0.01 and 1 (because of Taylor Series expansion)
- Left-shift the entries for lower-order functions
- Change time-step to zoom-in or zoom-out
- Get numerical values from spread sheet
- Copy and paste last row to extend spread sheet for more time rows (also adjust display range)

Final Remarks

- **Works with real, complex, multiple roots, pole-zero canceled roots, and right-hand plane zeros**
- **Response fast even though spread-sheet implementation is based on inefficient storage**
- **Recursive routine (slide 12) can be done <u>in-place</u> for better storage efficiency in other programming applications**
- **Display shows changes as coefficients are modified**
- **Display diverges if Laplace Transform close-form response diverges**

IBIS Summit
Downloads and References

- www.ibis.org/summits/nov15a/
  - ross2.xls (time-response utility)
  - ross2.pdf (this presentation for instructions, examples)
- www.ibis.org/summits/may11/
  - ross3.pdf, "Continuous and Discrete Modeling for IBIS-AMI" (gives theoretical background for both differential and difference equations)
  - ross2.pdf, "T-Coils and Bridged-T Networks" (gives general T-coil derivations)