Continuous and Discrete Modeling for IBIS-AMI

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Problem and Traditional Methods

- Relate, for example, tap coefficients to continuous function for IBIS-AMI
- Laplace Transform to/from Z Transform
  - Closed form tables
  - Bilinear transformation approximation commonly used by substituting for s or z:

$$z = e^{st} \cong \frac{1+sT/2}{1-sT/2},\ s = \frac{1}{T}\ln(z) \cong \left(\frac{2}{T}\right)\frac{z-1}{z+1}$$

- Exact solution given here

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References

- B. Ross, "Taylor Series Duality," *Proc. 7th IEEE Workshop of Signal Propagation on Interconnects*, Siena, Italy, May 11-14, 2003, pp. 97-100.
  - http://www.teraspeed.com
    - Full equations in paper
  - http://tinyurl.com/3pz8ec2
    - (presentation – temporary)

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Special Notion - Equations (1)-(4)

Laplace Transform	$X(s) = \dfrac{a_{n-1}s^{n-1} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_0}$,
Differential Equation	$x^n(t) + b_{n-1}x^{n-1}(t) + \cdots + b_0 x(t) = 0$ initial conditions, $x(0), \cdots, x^{n-1}(0)$,
Difference Equation	$x_n(t) + d_{n-1}x_{n-1}(t) + \cdots + d_0 x_0(t) = 0$ initial conditions, $x_0(0), \cdots, x_{n-1}(0)$,
Z Transform	$Z(z) = \dfrac{z(c_{n-1}z^{n-1} + \cdots + c_0)}{z^n + d_{n-1}z^{n-1} + \cdots + d_0}$.

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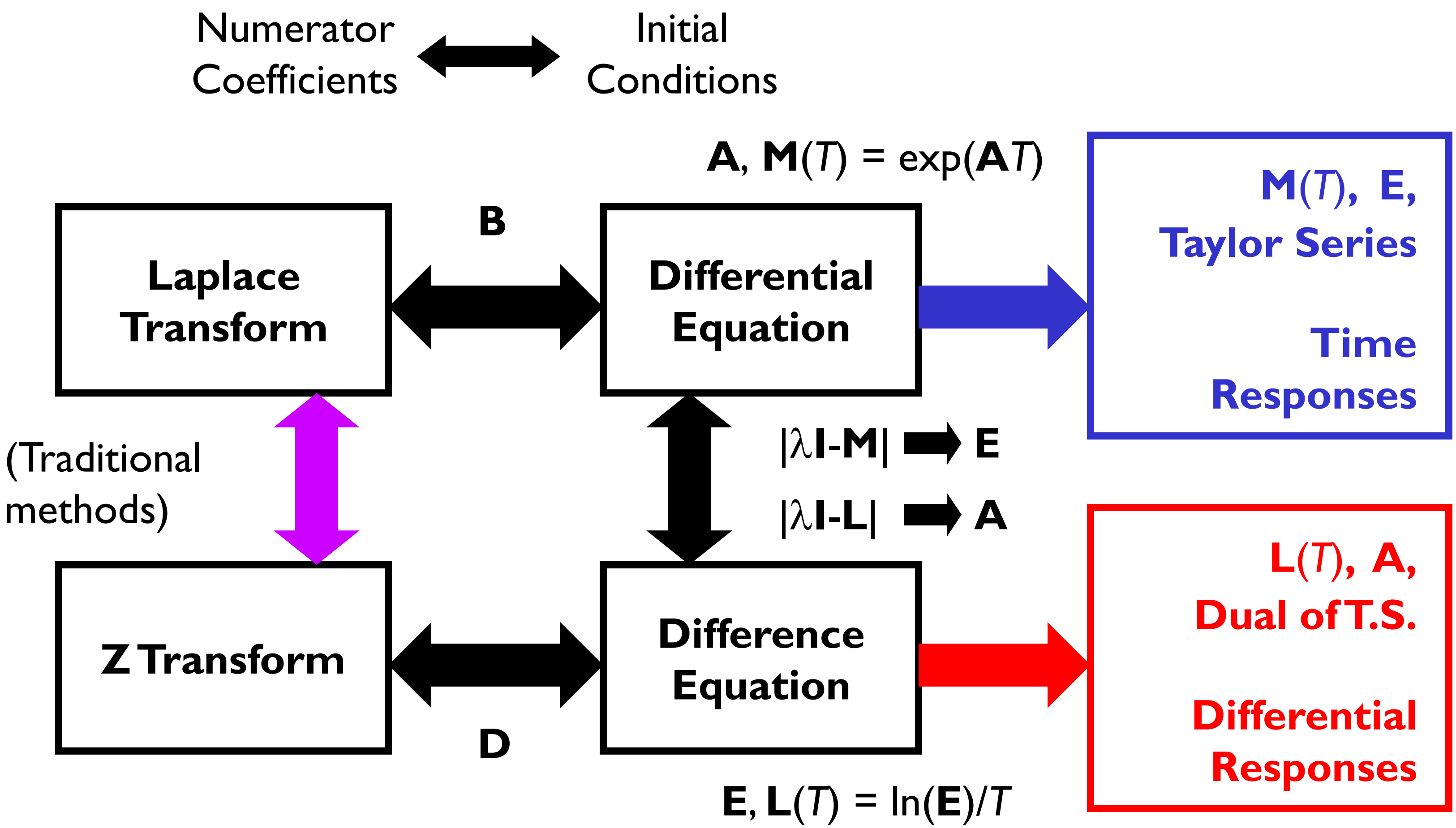
Conversions and Responses (5)-(26)



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Differential Eqn's (5)-(8)

$\mathbf{x}(t) = [x^0(t), x^1(t), \cdots, x^{n-1}(t)]^T,$

$\mathbf{a} = [a_{n-1}, \cdots, a_0]^T,$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ b_{n-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_2 & b_3 & \cdots & 1 & 0 \\ b_1 & b_2 & \cdots & b_{n-1} & 1 \end{bmatrix}$$

$\mathbf{a} = \mathbf{Bx}(0)$

Difference Eqn's (9)-(12)

$\mathbf{z}(t) = [x_0(t), x_1(t), \cdots, x_{n-1}(t)]^T$

$\mathbf{c} = [c_{n-1} \cdots c_0]^T$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ d_{n-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_2 & d_3 & \cdots & 1 & 0 \\ d_1 & d_2 & \cdots & d_{n-1} & 1 \end{bmatrix}$$

$\mathbf{c} = \mathbf{Dz}(0)$

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Differential Eqn's (13)-(16)

$d\mathbf{x}(t)/dt = \mathbf{A}\mathbf{x}(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & \cdots & -b_{n-1} \end{bmatrix}$$

$\mathbf{x}(t+T) = \mathbf{M}\mathbf{x}(t)$

$\mathbf{M} = \exp(\mathbf{A}T)$

Difference Eqn's (18)-(21)

$\mathbf{z}(t+T) = \mathbf{E}\mathbf{z}(t)$

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & \cdots & -d_{n-1} \end{bmatrix}$$

$d\mathbf{z}(t)/dt = \mathbf{L}\mathbf{z}(t)$

$\mathbf{E} = \exp(\mathbf{L}T)$

$\mathbf{L} = \ln(\mathbf{E})/T$

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Differential Eqn (17)

Difference Eqn (22)

$|\lambda\mathbf{I}\text{-}\mathbf{M}| =$

$$\lambda^n + d_{n-1}\lambda^{n-1} + \cdots + d_0 = 0$$

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & \cdots & -d_{n-1} \end{bmatrix}$$

$|\lambda\mathbf{I}\text{-}\mathbf{L}| =$

$$\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_0 = 0$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & \cdots & -b_{n-1} \end{bmatrix}$$

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Characteristic Equation

- Cayley-Hamilton Theorem – a matrix satisfies its own characteristic equation
- Computation of characteristic equations:
  - Based on built in mathematical functions
  - Or based on calculating traces (sum of diagonal terms) of powers of **M** or **L**
  - Further simplifications possible (outside scope of this presentation)

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Taylor Series Eqn's (23)-(24)

$$x_{j+1}{}^{i}(t) = \sum_{k=0}^{p} x_j{}^{i+k}(t)\alpha_k$$

$$\alpha_k = T^k / k!$$

Dual of T.S. Eqn (25)-(26)

$$x_j{}^{i+1}(t) = \sum_{k=0}^{q} x_{j+k}{}^{i}(t)\beta_k$$

$$\beta_0 = -\frac{1}{T}\sum_{i=1}^{q} 1/i$$

$$\beta_k = \frac{(-1)^{k-1}}{T}\binom{q}{k}\Big/k$$

$$\binom{q}{k} = \frac{q!}{k!(q-k)!} = \binom{q}{q-k}$$

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Recursive Taylor Series (Repeat b and c)

a) Initialize: i = 1, … , n-1 (**B**)

$$x(0) = a_{n-1} \qquad x^i(0) = a_{n-1-i} - \sum_{j=0}^{i-1} b_{n-i-j}\, x^j(0)$$

b) Extend: i = n, … , p (**A**)

$$x^i(t) = -\sum_{j=0}^{n-1} b_j\, x^{i-n-j}(t)$$

c) Next time step: i = 0, … , n-1 (Taylor series)

$$x^i(t+T) = \sum_{j=i}^{p} x^j(t) \frac{T^{j-i}}{(j-i)!}$$

R. I. Ross, "Evaluating the Transient Response of a Network Function," *Proc. IEEE*, vol.55, pp. 615-616, May 1967

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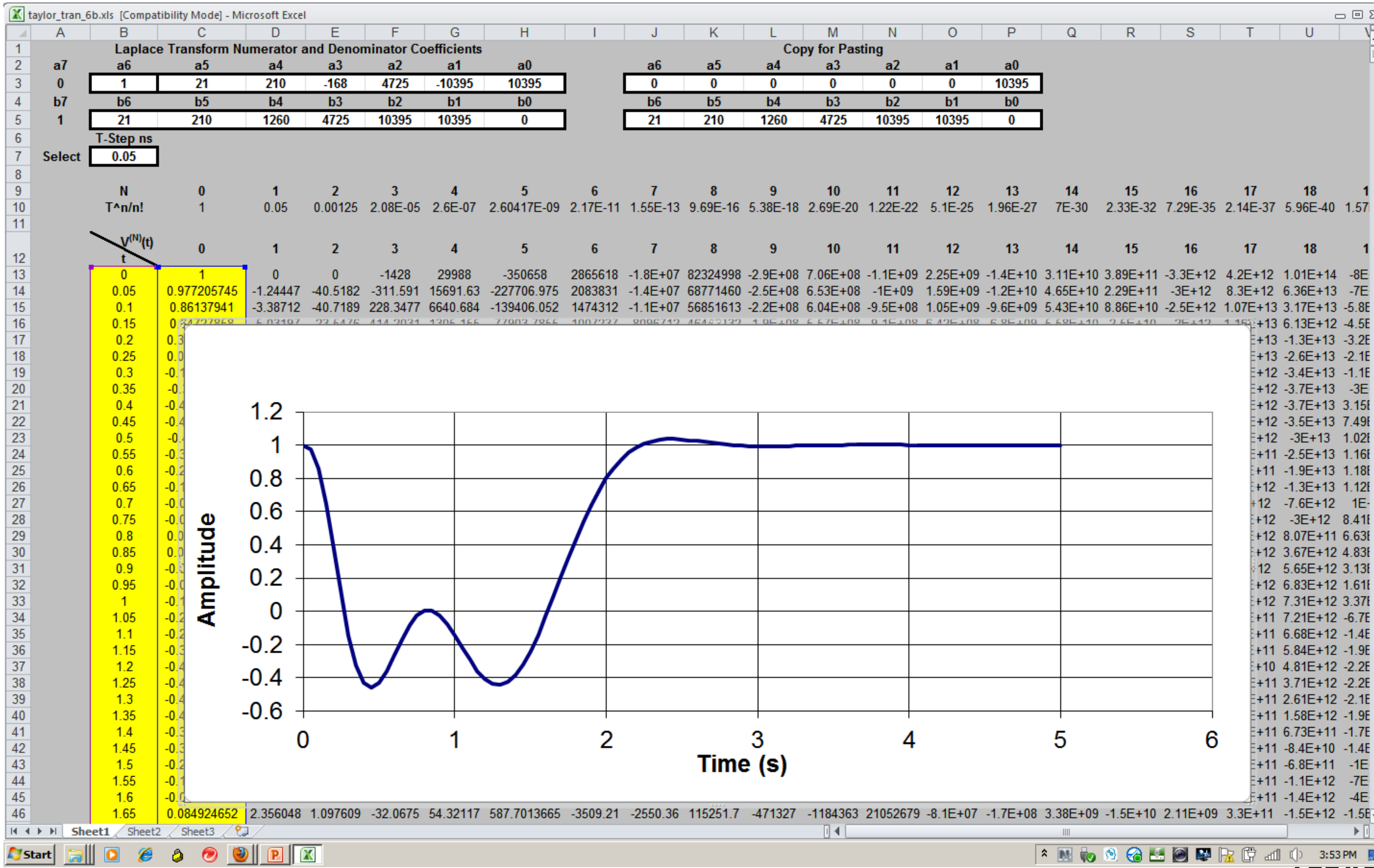
Excel T.S. Implementation



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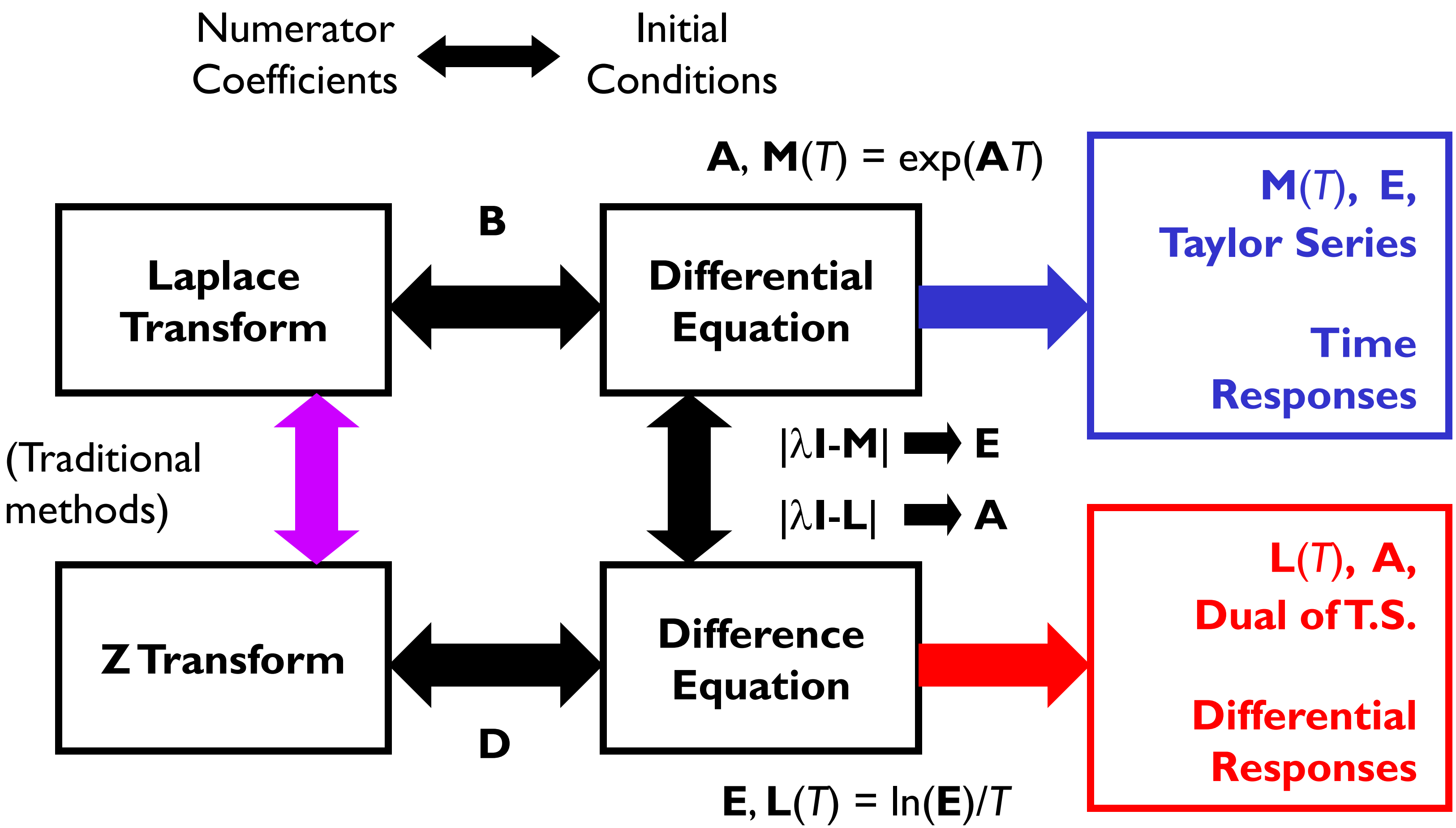
Conversions and Responses (5)-(26)



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Conclusions

- Recursive Taylor Series
  - In place, if needed
  - No special handing for multiple or complex poles as with partial fraction expansions
  - Embedded software and spread sheets
  - Good convergence with Taylor Series
- **M** or **L** can be calculated by functions of matrices can be used for exact continuous and discrete system conversions

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Background - Symmetry (State Transition/Logarithmic Matrices)

Difference (Time) $\mathbf{x}(t)$, $\mathbf{M}(T)$ →

Differential

$\mathbf{z}(t)$, $\mathbf{L}(T)$ ↓

$$
\begin{matrix}
x_0^0 & x_1^0 & \cdots & x_{n-1}^0 & x_n^0 & \cdots & x_u^0 \\
x_0^1 & x_1^1 & \cdots & x_{n-1}^1 & x_n^1 & \cdots & x_u^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} & \cdots & x_u^{n-1} \\
x_0^n & x_1^n & \cdots & x_{n-1}^n & x_n^n & \cdots & x_u^n \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
x_0^u & x_1^u & \cdots & x_{n-1}^u & x_n^u & \cdots & x_u^u
\end{matrix}
$$

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Transpose Symmetry (Difference/Differential Equation)

Difference (Time) **E** →

Differential

A ↓

$$
\begin{matrix}
x_0^0 & x_1^0 & \cdots & x_{n-1}^0 & x_n^0 & \cdots & x_u^0 \\
x_0^1 & x_1^1 & \cdots & x_{n-1}^1 & x_n^1 & \cdots & x_u^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} & \cdots & x_u^{n-1} \\
x_0^n & x_1^n & \cdots & x_{n-1}^n & x_n^n & \cdots & x_u^n \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
x_0^u & x_1^u & \cdots & x_{n-1}^u & x_n^u & \cdots & x_u^u
\end{matrix}
$$

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Transpose Symmetry (Taylor Series/Dual Responses)

Taylor Series

Difference (Time)

E

Differential

Dual of T.S.

A

$$
\begin{array}{cccc|ccc}
x_0^0 & x_1^0 & \cdots & x_{n-1}^0 & x_n^0 & \cdots & x_u^0 \\
x_0^1 & x_1^1 & \cdots & x_{n-1}^1 & x_n^1 & \cdots & x_u^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} & \cdots & x_u^{n-1} \\
\hline
x_0^n & x_1^n & \cdots & x_{n-1}^n & x_n^n & \cdots & x_u^n \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
x_0^u & x_1^u & \cdots & x_{n-1}^u & x_n^u & \cdots & x_u^u
\end{array}
$$

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Taylor Series and Dual of Taylor Series Derivations

- Taylor Series from State Transition Matrix
  - $\mathbf{x}(t+T) = \mathbf{M}(T)\mathbf{x}(t) = (\mathbf{I} + \mathbf{A}T + \mathbf{A}^2T^2/2! \ldots)\mathbf{x}(t)$
  - $\mathbf{A}^k\mathbf{x}(t)$ is $\boldsymbol{k}$-th derivative of $\mathbf{x}(t)$
- Dual of T.S. from Natural Logarithm Matrix
  - $d\mathbf{z}(t)/dt = \mathbf{L}(T)\mathbf{z}(t) = -[(\mathbf{I} - \mathbf{E}) + (\mathbf{I} - \mathbf{E})^2/2 + (\mathbf{I} - \mathbf{E})^3/3 + \ldots]\mathbf{z}(t)/T$
  - $\mathbf{E}^k\mathbf{z}(t)$ is the $\boldsymbol{k}$-th shifted sample of $\mathbf{z}(t)$
  - Collect the terms for each power of **E**

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Logarithmic Expansion: Binomial Series & Pascal Triangle Reduction

$q,i \downarrow \;/\; k \rightarrow$	factor $\downarrow$	0	1	2	3	4	5
0	–	–					
1	$\binom{q}{k}/1$	1/1	1/1				$\binom{q}{k}/i$
2	$\binom{q}{k}/2$	1/2	2/2	1/2			
3	$\binom{q}{k}/3$	1/3	3/3	3/3	1/3		
4	$\binom{q}{k}/4$	1/4	4/4	6/4	4/4	1/4	
5	$\binom{q}{k}/5$	1/5	5/5	10/5	10/5	5/5	1/5

$q,i \downarrow \;/\; k \rightarrow$	0	1	2	3	4	5
factor $\rightarrow$	$\sum_{i=1}^{q} 1/i$	$\binom{q}{1}/1$	$\binom{q}{2}/2$	$\binom{q}{3}/3$	$\binom{q}{4}/4$	$\binom{q}{5}/5$
0	–					
1	1/1	1/1				$\binom{q}{k}/k$
2	3/2	2/1	1/2			
3	11/6	3/1	3/2	1/3		
4	25/12	4/1	6/2	4/3	1/4	
5	137/60	5/1	10/2	10/3	5/4	1/5

$$\beta_k = \frac{(-1)^{k-1}}{T} \sum_{i=k}^{q} \binom{q}{k} \Big/ i, \quad k > 0$$

$$\beta_k = \frac{(-1)^{k-1}}{T} \binom{q}{k} \Big/ k, \quad k > 0$$

Correction: replace q with i

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Example: sin($10\pi t$), Some Original & Scaled Terms

- Scaling by weighting samples: $y_i = \exp(-\gamma t) x_i$
  -

k	$T\beta_k$	Scaled $T\beta_k$ (41 terms)
0	-3.14586	-3.14586
1	40.	12.8867
9	3.04e+7	1135.93 (maximum value)
20	-6.89e+9	-1.00 (set by scaling)
30	-2.83e+7	-4.93801e-8
40	-0.025	-5.26270e-22

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Example: sin($10\pi t$), Last 1/2 Cycle, 48-th Derivative

- 0.0 <= t <= 1.0, T = 0.02, 51 samples

Function	Exact	Dual T.S. (error bold)
sin[45]	0.0000**00**	**-1.08745e-5**
sin[46]	-0.587**785**	-0.587**844**
sin[47]	-0.951**057**	-0.951**141**
sin[48]	-0.951**057**	-0.951**134**
sin[49]	-0.587**785**	-0.587**827**
sin[50]	0.0000**00**	**1.08745e-5**

– (All other iterative methods are "Exact")

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Example: sin($10\pi t$), Last 1/2 Cycle, 49-th Derivative

- $0.0 <= t <= 1.0$, $T = 0.02$, 51 samples

Function	Exact	Dual T.S. (error bold)
cos[45]	-1.000**00**	-1.000**90**
cos[46]	-0.8090**17**	-0.8090**81**
cos[47]	-0.3090**17**	-0.3090**33**
cos[48]	0.3090**17**	0.3090**55**
cos[49]	0.8090**17**	0.8090**94**
cos[50]	1.000**00**	1.000**90**

– (All other iterative methods are "Exact")

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Conclusions

- Exact transformations
  - Practical modeling applications
  - Common routines for both domains
- Taylor Series & Binomial Series “duality”
  - Accurate with scaling
  - But not as accurate and stable as other iterative methods

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